Classification of irregular free boundary points for non-divergence type equations with discontinuous coefficients

CLASSIFICATION OF IRREGULAR FREE BOUNDARY POINTS FOR NON-DIVERGENCE TYPE EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

SERENA DIPIERRO, ARAM KARAKHANYAN, AND ENRICO VALDINOCI

Abstract. We provide an integral estimate for a non-divergence (non-variational) form second order elliptic equation $a_{ij}u_{ij} = u^p$, $u \geq 0$, $p \in [0,1)$, with bounded discontinuous coefficients a_{ij} having small BMO norm. We consider the simplest discontinuity of the form $x \otimes x|x|^{-2}$ at the origin. As an application we show that the free boundary corresponding to the obstacle problem (i.e. when $p = 0$) cannot be smooth at the points of discontinuity of $a_{ij}(x)$.

To implement our construction, an integral estimate and a scale invariance will provide the homogeneity of the blow-up sequences, which then can be classified using ODE arguments.

1. Introduction

One of the main distinctions in the field of partial differential equations consists in the difference between equations "in divergence form" and those "in non-divergence form". While the first ones naturally admit a variational formulation and can be dealt with by energy methods, the second ones usually require different – and perhaps more sophisticated – techniques (see e.g. [T82] for a detailed discussion), often in combination with viscosity methods.

We refer to [K07, C08] and the references therein for throughout presentations of similarities and differences between equations in divergence and non-divergence form.

A similar distinction between divergence and non-divergence structure occurs in the field of free boundary problems. As a matter of fact, free boundary problems whose partial differential equation is in divergence form often enjoy a special feature given by the so-called "monotonicity formulas": namely, the energy functional, or a suitable variational integral, possesses a natural monotonicity property with respect to some geometric quantity (typically, a functional defined on balls of radius r turns out to be monotone in r).

This type of monotonicity property is, in a sense, geometrically motivated, since it may be seen somehow as an offspring of classical monotonicity formulas arising in the theory of minimal surfaces and geometric flows. In addition, combined with the natural scaling of the problem, a monotonicity formula is often very useful in proving uniqueness of blow-up solutions, classification results and regularity theorems.

Viceversa, problems which do not enjoy monotonicity formulas (or for which a monotonicity formula is not known) may turn out to be considerably harder to deal with, and proving (or disproving) a strong regularity theory is a natural, important and often very challenging question (see e.g. [CS05, PSU12] for further discussions on monotonicity formulas).

In this paper we consider the free boundary problem

$$\mathcal{L}v := a_{ij}v_{ij} = v^p \text{ in } B_1, \quad v \geq 0, \tag{1.1}$$

with $p \in (0,1)$. We will also deal with the case $p = 0$ using the notation that identifies v to the power zero with the characteristic function $\chi_{\{v>0\}}$.

Problems of this type often arise in real world phenomena. For instance, in the study of the spread of biological populations one studies the problem

$$\operatorname{div}(a\nabla(u^m)) + f(x)u + b \cdot \nabla(u^m) = 0 \tag{1.2}$$

where $u : \mathbb{R}^n \to [0,+\infty)$ represents the density of the population, $a : \mathbb{R}^n \to \mathrm{Mat}(n \times n)$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ represents a drift term. Here, $m > 1$, $a(x)$ is a positive definite matrix (with entries $a_{ij}(x)$) and $f : \mathbb{R}^n \to \mathbb{R}$ takes into account the influence of the environment on the population, see [S83].

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It is convenient to reformulate the problem in terms of the auxiliary function $v := u^m$ and write (1.2) as

$$\mathrm{div}(a\nabla v) + f(x)v^{\frac{1}{m}} + b \cdot \nabla v = 0.$$

Notice that this boils down to the equation in (1.1) when $m = 1/p$, $f \equiv -1$ and $b = (b_1, \dots, b_n)$ with $b_i = \partial_j a_{ij}$.

Moreover, equations in non-divergence form arise naturally from probabilistic considerations, for instance, as the infinitesimal generators of anisotropic random walks, see e.g. Section 2.1.3 in [C08].

Furthermore, when a in (1.1) is the identity matrix, the problem reduces to the classical one in [AP86], which, in turn, as $p \to 0$ recovers the exemplary free boundary problem in [C77].

Our objective in the present paper is to study the behavior of the solution v of (1.1) near the free boundary points $x \in \partial\{v > 0\}$ at which the matrix $a_{ij}(x)$ is discontinuous. A model example of this sort in 2D is

$$\Delta v + \varepsilon \left(\frac{x_1^2}{|x|^2} v_{22} - \frac{2x_1x_2}{|x|^2} v_{12} + \frac{x_2^2}{|x|^2} v_{11} \right) = v^p \tag{1.3}$$

where ε is a small constant and $p \in [0, 1)$ (here, we are using the standard notation $x = (x_1, x_2) \in \mathbb{R}^2$ and $v_{ij} = \partial^2_{ij} v$).

One can also write equation (1.3) in the equivalent form

$$\mathrm{div}(a\nabla v) + b \cdot \nabla v = v^p$$

where

$$a(x) := \begin{pmatrix} 1 + \frac{\varepsilon x_2^2}{|x|^2} & -\frac{\varepsilon x_1 x_2}{|x|^2} \\ -\frac{\varepsilon x_1 x_2}{|x|^2} & 1 + \frac{\varepsilon x_1^2}{|x|^2} \end{pmatrix} \tag{1.4}$$

and

$$b = (b^1, b^2), \quad b^j = -\sum_i \partial_i(a_{ij}), \quad |b| \sim \frac{1}{|x|}.$$

We observe that the quadratic form

$$a_{ij}\xi_i\xi_j = |\xi|^2 + \frac{\varepsilon}{|x|^2}\left((x_1\xi_2)^2 + (x_2\xi_1)^2 - 2x_1x_2\xi_1\xi_2\right) = |\xi|^2 + \frac{\varepsilon}{|x|^2}(x_1\xi_2 - x_2\xi_1)^2$$

is positive definite and a_{ij} are discontinuous at the origin.

More generally, we can assume that the diffusion matrix a has the form

$$a_{ij}(x) = h_{ij}(x) + b_{ij}(x) \tag{1.5}$$

where h_{ij} is a homogeneous function of degree zero and for any point $x_0 \in \mathbb{R}^n$ we have that

$$|b_{ij}(x) - \delta_{ij}| \le \omega(|x - x_0|),$$

with

$$\int_0^\delta \frac{\omega(t)}{t} dt < +\infty,$$

for some $\delta > 0$. Roughly speaking, in (1.5), the terms b_{ij} and h_{ij} represent the continuous and the discontinuous parts of a_{ij}, respectively.

Throughout this paper we will assume that the operator satisfies the following conditions:

- **(H1)** the entries of the matrix a_{lm} are bounded measurable functions, and the matrix is uniformly elliptic, i.e. there exist two positive constants λ and Λ such that
$$\lambda|\xi|^2 \le a_{lm}(x)\xi_l\xi_m \le \Lambda|\xi|^2, \quad \forall x \in B_1,$$
- **(H2)** the coefficients $a_{lm}(x)$ have small BMO norm, namely
$$\sup_{0<r\le R} \sup_{x\in\mathbb{R}^n} ⨍_{B_r(x)} \left| a_{lm}(y) - ⨍_{B_r(x)} a_{lm} \right| dy = \delta(R) < +\infty,$$
where $\delta(R) > 0$ is a small constant.
- **(H3)** the matrix a_{ij} has at least one discontinuity at $x_0 \in \mathbb{R}^n$ such that $a_{ij}(x)$ is rotational invariant at x_0 and homogeneous of degree one.

In this setting, the problem in (1.1) admits a solution, as given by the following result:

Theorem 1.1. *Let $g \in W^{2,\infty}(B_1) \cap C(\overline{B_1})$, with $g \geq 0$. Then, there exists a nonnegative function v such that $v - g \in W^{2,q}(B_1) \cap W_0^{1,q}(B_1)$, for all $1 < q < +\infty$, and v solves (1.1).*

From the technical point of view, concerning the assumptions on the coefficients a_{ij}, we notice that the function $x_i x_j |x|^{-2} \notin VMO$ for any i and j. However, if ε is sufficiently small then (**H2**) holds with $\delta(R) \leq C\varepsilon$, where C is a dimensional constant. Consequently, we can apply the $W^{2,q}$ estimates from [CFL93] to establish the existence and optimal growth of the solutions. As a matter of fact, setting

$$\beta = \frac{2}{1-p}, \tag{1.6}$$

we can bound the growth from the free boundary according to the following result:

Theorem 1.2. *Let $v \geq 0$ be a bounded weak solution of (1.1) in B_1. Then there exists a constant $M > 0$, possibly depending on $\|v\|_{L^\infty(B_1)}$, such that, for each $\bar{x} \in B_{\frac{1}{2}} \cap \partial\{v > 0\}$ and any $x \in B_{\frac{1}{4}}(\bar{x})$, it holds that $v(x) \leq M\,|x - \bar{x}|^\beta$.*

We remark that the problem in (1.1) has a natural scale invariance: for this, it is useful to define

$$v_r(x) := \frac{v(x_0 + rx)}{r^\beta}$$

with β as in (1.6). We notice indeed that v_r is also a solution of (1.1). We will show that, up to a subsequence, these blow-up functions approach a blow-up limit.

We say that v is non-degenerate at $x_0 \in \partial\{v > 0\}$ if there exists a sequence of positive numbers $r_k \to 0$ such that the corresponding blow-up limit is not identically zero.

A cornerstone of our analysis is a uniform integral estimate. The result that we obtain is the following:

Theorem 1.3. *Let v be a strong solution of (1.1) in $B_1 \subset \mathbb{R}^2$, with a_{ij} as in (1.4). Assume that $0 \in \partial\{v > 0\}$ and v is non-degenerate at 0. Then*

$$\int_{B_{1/2}} \left(\beta\, \frac{v(x)}{|x|^\beta} - \frac{\partial_r v(x)}{|x|^{\beta-1}} \right)^2 \frac{dx}{|x|^2} \leq \widetilde{C}, \tag{1.7}$$

for some $\widetilde{C} > 0$ possibly depending on $\|v\|_{L^\infty(B_1)}$.

In this framework, the integral estimate in (1.7), combined with the scale invariance, implies that the blow-up limits are homogeneous, as described in the following result:

Theorem 1.4. *Let v be a strong solution of (1.1) in B_1, with a_{ij} as in (1.4). Assume that $0 \in \partial\{v > 0\}$ and v is non-degenerate at 0. Then any blow-up sequence at 0 has a converging subsequence such that the limit is a homogeneous function of degree $\beta = \frac{2}{1-p}$.*

This result will in turn play a special role for the classification of global solutions. Roughly speaking, the homogeneity property, an appropriate use of polar coordinates and explicit methods borrowed from the theory of ordinary differential equations lead to a classification of solutions growing in a non-degenerate way from a smooth free boundary. This classification and the analysis of the blow-up limits will be the main ingredients for the analysis of irregular free boundary points, as explained in the following result:

Theorem 1.5. *Let $n = 2$, $\mathcal{L}$ be as in (1.1) and a_{ij} as in (1.4), with $|\varepsilon|$ sufficiently small. Let v be a solution of (1.1) in B_1 with $p = 0$. Assume that $0 \in \partial\{v > 0\}$ and that v is non-degenerate at 0. Then $\partial\{v > 0\}$ cannot be differentiable at the origin.*

The paper is organized as follows: in Section 2 we establish the existence of a strong solution of (1.1) in the unit ball B_1 and thus prove Theorem 1.1. Next, using a dyadic scaling argument, we prove that a solution $v(x)$ grows away from the free boundary $\partial\{v > 0\}$ as $[\mathrm{dist}(x, \partial\{v > 0\})]^\beta$. This is contained in Section 3, which will provide the proof of Theorem 1.2. Our main technical tool, which is the uniform integral bound in Theorem 1.3, is established in Section 4. To this goal, we use some computations based on the ideas of Joel Spruck [S83]. Section 4 also contains the proof of Theorem 1.4, which fully relies on the integral estimate in (1.7). Finally, in Section 5 we show that the free boundary cannot be regular at the free boundary points where a_{ij} suffers a discontinuity satisfying (**H3**), thus completing the proof of our main result in Theorem 1.5.

2. Existence of solutions

In this section, we give the proof of the existence result in Theorem 1.1.

Proof of Theorem 1.1. The proof is based on a classical penalization argument. The case of the obstacle problem, corresponding to $p = 0$, is treated in [BT14]. Our proof is similar, but we will sketch it for the reader's convenience since unlike [BT14] our coefficients are not in VMO. In fact, for our case $p \in (0, 1)$ the proof is shorter since for $p > 0$ the penalization function ϕ_ε (see below) is continuous at the origin. Hence, by a customary compactness argument, we deduce that the limit of the penalized problem is a solution of (1.1) a.e. Therefore, we only need to establish uniform estimates for the penalized problem (2.5). The details of the proof go as follows.

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \eta \subset B_1$, $\eta \geq 0$ and $\int_{B_1} \eta = 1$. Let $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$. Then η_ϵ is a standard mollifier. Set $a_{ij}^\epsilon := a_{ij} * \eta_\epsilon$ and $g_\epsilon := g * \eta_\epsilon$. Furthermore, let $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ be a family of functions with the following properties

$$\begin{aligned} & 0 \leq \phi_\epsilon(s) \leq 1, \\ & \phi_\epsilon(s) = 0 \quad \text{if } s \leq 0, \\ & \phi_\epsilon(s) = s^p \quad \text{if } s \geq \epsilon, \\ & \phi_\epsilon(s) \text{ is monotone increasing,} \\ \text{and} \qquad & \phi_\epsilon \in C^\infty(\mathbb{R}). \end{aligned}$$

Then, there exists a classical solution v^ϵ to the following Dirichlet problem

$$\begin{cases} a_{ij}^\epsilon(x)\partial_{ij} v^\epsilon(x) = \phi_\epsilon(v^\epsilon(x)) \text{ in } B_1, \\ v^\epsilon(x) = g_\epsilon(x) \text{ on } \partial B_1. \end{cases}$$

Now, for every $t \in [0, 1]$, we consider the penalized problem

$$\begin{cases} a_{ij}^\epsilon(x)\partial_{ij} v_t^\epsilon(x) = t\phi_\epsilon(v_t^\epsilon(x)) \text{ in } B_1, \\ v_t^\epsilon(x) = g_\epsilon(x) \text{ on } \partial B_1. \end{cases} \tag{2.1}$$

We set

$$I := \{t \in [0, 1] \text{ s.t. (2.1) has a solution}\}$$

and we claim that

$$I \text{ is open.} \tag{2.2}$$

Note that $a_{ij}^\epsilon(x)\partial_{ij} v_t^\epsilon(x) \geq 0$, hence by the maximum principle $0 \leq v_t^\epsilon(x) \leq \|g_\epsilon\|_\infty$. For any $t \in [0, 1]$, we consider the operator $A_t u := a_{ij}u_{ij} - t\phi_\epsilon(u)$. Then the Fréchet derivative of A_t is

$$DA_t h = a_{ij}h_{ij} - t\phi'_\epsilon(u)h.$$

Thus the derivative operator has the form

$$DA_t h = a_{ij}h_{ij} + tc(x)h, \quad \text{with } c(x) \leq 0$$

since, by construction, ϕ_ϵ is monotone increasing. Applying the Schauder theory in Chapter 6 of [GT98], we conclude that for any $f \in C^\alpha$ and $g \in C^{2,\alpha}(\overline{B_1})$ there exists a solution w^ϵ of

$$\begin{cases} DA_t w^\epsilon = f \text{ in } B_1, \\ w^\epsilon(x) = g_\epsilon(x) \text{ on } \partial B_1. \end{cases} \tag{2.3}$$

This implies that $DA_t : C^{2,\alpha}(\overline{B_1}) \to C^{2,\alpha}(\overline{B_1}) \oplus C^\alpha(\overline{\partial B_1})$ is surjective. By the maximum principle (recall that $c(x) = -\phi'_\epsilon(v_t^\epsilon) \leq 0$) DA_t is also injective. Therefore, DA_t is invertible, which establishes (2.2).

Now we show that

$$I \text{ is closed.} \tag{2.4}$$

To this aim, we first observe that, from the Sobolev embedding, we have that $\|v_t^\epsilon\|_{C^{1,\alpha}} \lesssim \|v_t^\epsilon\|_{W^{2,q}}$. Consequently, applying the Schauder estimates in Chapter 6 of [GT98], we obtain that

$$\|v_t^\epsilon\|_{C^{4,\alpha}} \leq C(\epsilon),$$

for some $C(\epsilon) > 0$, independently of t. Thus if $I \ni t_k \to t_0$ then from Arzela-Ascoli theorem it follows that $v^\epsilon_{t_k} \to v^\epsilon_{t_0}$ in $C^{4,\alpha}(\overline{B_1})$ and $v^\epsilon_{t_0}$ solves the corresponding problem (2.1), thus proving (2.4).

Now, from (2.2) and (2.4), we deduce that a solution of (2.1) exists for all $t \in [0,1]$. By Theorem 4.2 in [CFL93], we have that

$$\|v^\epsilon_t\|_{W^{2,q}(B_1)} \le C, \quad q > 1, \tag{2.5}$$

uniformly in ϵ because a^ϵ_{ij} verifies (**H1**)-(**H3**). □

3. Optimal growth from the free boundary

Let $x_0 \in \partial\{v > 0\} \cap B_1$ and consider the scaled function

$$v_r(x) := \frac{v(x_0 + rx)}{r^\beta}, \qquad r > 0.$$

We remark that if the inequality

$$v(x) \le C|x - x_0|^\beta \tag{3.1}$$

holds in some neighborhood of x_0, for some constant $C > 0$ and β as in (1.6), then v_r is uniformly bounded as $r \to 0$.

So, we show that the growth control in (3.1) is indeed satisfied for bounded solutions of (1.1). The result that we have is the following:

Proposition 3.1. *Let $u \ge 0$ be a weak solution of (1.1) in B_1 such that*

$$0 \le u(x) \le M$$

for some constant $M > 0$. Then there exists a constant $C > 0$ such that for each $x \in B_{\frac{1}{2}} \cap \partial\{v > 0\}$ there holds

$$S(k+1, x) \le \max\left\{\frac{CM}{2^{\beta k}}, \frac{1}{2}S(k, x)\right\} \tag{3.2}$$

where $S(k, x) := \sup_{B_{2^{-k}}(x)} u$.

Remark 3.2. It is well known that the estimate in Proposition 3.1 implies the desired growth rate in (3.1).

Proof of Proposition 3.1. We use a dyadic scaling argument. Suppose that the claim in Proposition 3.1 fails, then there exists a sequence of integers k_i, and points $x_i \in B_{\frac{1}{2}} \cap \partial\{v > 0\}$ such that

$$S(k_i + 1, x_i) > \max\left\{\frac{iM}{2^{\beta k_i}}, \frac{1}{2}S(k_i, x_i)\right\}. \tag{3.3}$$

We introduce the scaled functions

$$u_i(x) := \frac{v(x_i + 2^{-k_i}x)}{S(k_i + 1)}, \tag{3.4}$$

where $S(\cdot)$ is a short notation for $S(\cdot, x_i)$. Then, we have that

$$\sup_{B_{\frac{1}{2}}} u_i = \frac{\sup_{B_{2^{-k_i+1}}(x_i)} u}{S(k_i + 1)} = 1, \tag{3.5}$$

and, from (3.3),

$$\sup_{B_1} u_i = \frac{\sup_{B_{2^{-k_i}}(x_i)} u}{S(k_i + 1)} = \frac{S(k_i)}{S(k_i + 1)} \le 2. \tag{3.6}$$

Furthermore, setting $r_i := 2^{-k_i}$, by a direct computation we see that

$$\begin{aligned}
\sum_{l,m} a_{lm}(x_i + xr_i)\partial_{lm} u_i(x) &= \frac{2^{-2k_i}}{S(k_i + 1)} u^p(x_i + xr_i) \\
&= \frac{2^{-2k_i} S^p(k_i + 1)}{S(k_i + 1)} u_i^p(x) \\
&= \frac{1}{2^{2k_i} S^{1-p}(k_i + 1)} u_i^p(x).
\end{aligned}$$

Notice also that (3.3) and (1.6) yield that

$$\begin{aligned} iM &\le 2^{\beta k_i} S(k_i+1) \\ &= \left(2^{2k_i} S^{\frac{2}{\beta}}(k_i+1)\right)^{\frac{\beta}{2}} \\ &= \left(2^{2k_i} S^{1-p}(k_i+1)\right)^{\frac{\beta}{2}}. \end{aligned}$$

Consequently, recalling (3.6), we have that

$$0 \le \sum_{l,m} a_{lm}(x_i + x r_i)\partial_{lm} u_i(x) \le \frac{u_i^p(x)}{(k_i M)^{\frac{2}{\beta}}} \le \frac{2^p}{(k_i M)^{\frac{2}{\beta}}} \to 0 \quad \text{as} \quad i\to\infty. \tag{3.7}$$

Let us define the sequence of matrices $A^i_{lm}(x) := a_{lm}(x_i + r_i x)$. Then $A^i(x)$ satisfies (**H1**). Observe that the change of variables $\xi = x_i + r_i x$ implies

$$⨍_{B_r(z)} A^i_{lm} = ⨍_{B_{rr_i}(x_i + r_i z)} a_{lm}.$$

Recalling that $x\in B_{\frac12}$, we see that

$$\sup_{0<r\le R}\sup_{z\in\mathbb{R}^n} ⨍_{B_r(z)} \left| A^i_{lm}(x) - ⨍_{B_r(z)} A^i_{lm}\right| dx = \sup_{0<r\le Rr_i}\sup_{y\in\mathbb{R}^n} ⨍_{B_r(y)} \left| a_{lm}(\xi) - ⨍_{B_r(y)} a_{lm}\right| d\xi \le \delta(Rr_i), \tag{3.8}$$

implying that (**H2**) is also satisfied for the matrices A^i.

Furthermore, in light of (3.7), we see that u_i solves the inequality

$$\left|\sum_{l,m} A^i_{lm}(x)\partial_{lm}u_i(x)\right| \le \frac{2^p}{(k_i M)^{\frac{2}{\beta}}} \to 0. \tag{3.9}$$

From (3.6), (3.8) and (3.9) it follows that we can apply Theorem 4.1 in [CFL93] to conclude that for any $q>1$ the following estimate holds uniformly in i

$$\|u_i\|_{W^{2,q}(B_\rho)} \le C(\rho, q) \tag{3.10}$$

where B_ρ is a fixed ball but with arbitrary radius $\rho>0$. Consequently, the sequence of strong solutions $\{u_i\}$ is bounded in $W^{2,q}_{loc}\cap L^\infty$. From Krylov-Safonov theorem it follows that for a subsequence, still denoted by u_i, we have that $u_i \to u$ in $B_{\frac34}$ uniformly. Thus $u_i(0)=0$ and (3.5) translates to the limit function u, namely we have

$$u(0)=0, \quad u(x)\ge 0, \quad \sup_{B_{\frac12}} u = 1, \quad \sup_{B_1} u \le 2.$$

On the other hand $A^i\to A^0$ a.e. and A^0 satisfies (**H1**)-(**H3**). In particular, $A^0_{lm}u_{lm}=0$ a.e. Hence, $u(0)=0$ and the strong maximum principle imply that $u\equiv 0$ which is in contradiction with $\sup_{B_{\frac12}} u = 1$ and the proof is complete. ☐

From Proposition 3.1 and Remark 3.2 we obtain Theorem 1.2, as desired.

4. Blow-up sequences and homogeneity

We want to show that, using a technique invented by J. Spruck in [S83], at the non degenerate free boundary points the blow-up is a homogeneous function of degree β. For a sequence of positive numbers $r_k\to 0$ and $x_0\in\partial\{v>0\}$, we consider the blow-up sequence

$$v_{r_k}(x) := \frac{v(x_0 + r_k x)}{r_k^\beta}. \tag{4.1}$$

From Theorem 1.2 we know that the sequence $\{v_{r_k}\}$ is bounded and solves equation (1.1) with a_{ij} satisfying (**H1**)-(**H3**). Thus, applying Theorem 4.1 in [CFL93], we conclude that $\{v_{r_k}\}$ is locally uniformly bounded in $W^{2,q}$ for any $q>1$. Then a customary compactness argument implies that there exists a subsequence $\{v_{k_i}\}$ and v_0, such that

$$v_{k_i} \to v_0 \text{ in } C^1_{loc}(\mathbb{R}^n). \tag{4.2}$$

The function v_0 is called a blow-up limit at x_0.

4.1. 2**D problems.** As customary, it is often useful to write solutions of partial differential equations in polar coordinates. In our case, we have the following result:

Lemma 4.1. *Let $\mathcal{L}$ be as in (1.1), with a_{ij} as in (1.4). Then*

$$\mathcal{L}v = \partial_{rr}v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial_{\theta\theta}v + \varepsilon\left(\frac{\partial_{\theta\theta}v}{r^2} + \frac{\partial_r v}{r}\right). \tag{4.3}$$

Proof. We will use polar coordinates r, θ and rewrite the partial derivatives as follows

$$\partial_{x_1} = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \quad \partial_{x_2} = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta. \tag{4.4}$$

By a straightforward computation we have that

$$\begin{aligned}
2x_1x_2\partial_{12}v = 2r^2\cos\theta\sin\theta\Big\{&\sin\theta\cos\theta\partial_{rr}v - \frac{\sin\theta\cos\theta}{r}\partial_r v + \frac{\cos^2\theta - \sin^2\theta}{r}\partial_{\theta r}v \\
&+ \frac{\sin^2\theta - \cos^2\theta}{r^2}\partial_\theta v - \frac{\sin\theta\cos\theta}{r^2}\partial_{\theta\theta}v\Big\},\\
x_2^2\partial_{11}v = r^2\sin^2\theta&\left\{\cos^2\theta\partial_{rr}v + \frac{\sin^2\theta}{r}\partial_r v - \frac{2\sin\theta\cos\theta}{r}\partial_{r\theta}v + \frac{2\sin\theta\cos\theta}{r^2}\partial_\theta v + \frac{\sin^2\theta}{r^2}\partial_{\theta\theta}v\right\},\\
x_1^2\partial_{22}v = r^2\cos^2\theta&\left\{\sin^2\theta\partial_{rr}v + \frac{\cos^2\theta}{r}\partial_r v + \frac{2\sin\theta\cos\theta}{r}\partial_{r\theta}v - \frac{2\sin\theta\cos\theta}{r^2}\partial_\theta v + \frac{\cos^2\theta}{r^2}\partial_{\theta\theta}v\right\}.
\end{aligned}$$

Combining these three identities and recognizing the terms we get that

$$\begin{aligned}
&\frac{1}{\varepsilon}(\mathcal{L}v - \Delta v)\\
= \;&\partial_{r\theta}\frac{2\sin\theta\cos\theta}{r}\left[\cos^2\theta - \sin^2\theta - \cos^2\theta + \sin^2\theta\right] + \partial_{\theta\theta}v\left[\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\right]\\
&+\partial_r v\frac{1}{r}\left[\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\right] + \partial_\theta v\frac{2\sin\theta\cos\theta}{r}\left[\sin^2\theta - \cos^2\theta - \sin^2\theta + \cos^2\theta\right]\\
= \;&\frac{\partial_{\theta\theta}v}{r^2} + \frac{\partial_r v}{r}.
\end{aligned}$$

Using this and the standard representation of the Laplacian in polar coordinates, the desired result follows. ∎

With this, we are in position of proving Theorem 1.3.

Proof of Theorem 1.3. We let $r := e^{-t}$ and $w(t,\theta) := \frac{v(r,\theta)}{r^\beta}$. Then we have

$$\begin{aligned}
\partial_\theta w &= \frac{\partial_\theta v}{r^\beta},\\
\partial_{\theta\theta}w &= \frac{\partial_{\theta\theta}v}{r^\beta},\\
\text{and} \qquad \partial_t w &= -\frac{\partial_r v}{r^{\beta-1}} + \beta w.
\end{aligned}$$

Plugging this into (4.3) we infer that

$$r^{\beta-2}((\partial_{tt}w - \partial_t w - \beta(\beta-1)w) + (\beta w - \partial_t w) + \partial_{\theta\theta}w) + \varepsilon\left(r^{\beta-2}\partial_{\theta\theta}w + r^{\beta-2}[\beta w - \partial_t w]\right) = r^{-\beta p}w^p.$$

This, after recalling that $\beta - 2 = -p\beta$, yields that

$$I_1 + \varepsilon I_2 = w^p, \tag{4.5}$$

where

$$I_1 := \partial_{tt}w - 2\partial_t w + \partial_{\theta\theta}w - \beta(\beta-2)w \quad \text{and} \quad I_2 := \partial_{\theta\theta}w + \beta w - \partial_t w.$$

Next, we multiply both sides of equation (4.5) by $\partial_t w$ and we integrate first over the unit circle and then in the interval $[T_1, T_2]$ to get that

$$\int_{T_1}^{T_2}\int_{\mathbb{S}^1} I_1\,\partial_t w + \varepsilon\int_{T_1}^{T_2}\int_{\mathbb{S}^1} I_2\,\partial_t w = \int_{T_1}^{T_2}\int_{\mathbb{S}^1} w^p\,\partial_t w. \tag{4.6}$$

Now we observe that

$$\begin{aligned}
\int_{T_1}^{T_2}\int_{\mathbb{S}^1} I_2\,\partial_t w &= -\int_{T_1}^{T_2}\int_{\mathbb{S}^1}(\partial_t w)^2 + \beta\int_{T_1}^{T_2}\int_{\mathbb{S}^1} w\,\partial_t w + \int_{T_1}^{T_2}\int_{\mathbb{S}^1}\partial_{\theta\theta} w\,\partial_t w\\
&= -\int_{T_1}^{T_2}\int_{\mathbb{S}^1}(\partial_t w)^2 + \beta\int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2} - \int_{T_1}^{T_2}\int_{\mathbb{S}^1}\partial_\theta w\,\partial_{r\theta} w \qquad (4.7)\\
&= -\int_{T_1}^{T_2}\int_{\mathbb{S}^1}(\partial_t w)^2 + \beta\int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2} - \int_{\mathbb{S}^1}\frac{(\partial_\theta w)^2}{2}\Big|_{T_1}^{T_2}.
\end{aligned}$$

Similarly,

$$\int_{T_1}^{T_2}\int_{\mathbb{S}^1} I_1\,\partial_t w = -2\int_{T_1}^{T_2}\int_{\mathbb{S}^1}(\partial_t w)^2 + \int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2} - \int_{\mathbb{S}^1}\frac{(\partial_\theta w)^2}{2}\Big|_{T_1}^{T_2} - \frac{\beta(\beta-2)}{2}\int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2}. \tag{4.8}$$

Moreover,

$$\int_{T_1}^{T_2}\int_{\mathbb{S}^1} w^p\,\partial_t w = \int_{\mathbb{S}^1}\frac{1}{p+1}w^{p+1}\Big|_{T_1}^{T_2}$$

So, plugging this, (4.7) and (4.8) into (4.6), we obtain that

$$\begin{aligned}
(\varepsilon+2)\int_{T_1}^{T_2}\int_{\mathbb{S}^1}(\partial_t w)^2 &= \varepsilon\left\{\beta\int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2} - \int_{\mathbb{S}^1}\frac{(\partial_\theta w)^2}{2}\Big|_{T_1}^{T_2}\right\} - \int_{\mathbb{S}^1}\frac{1}{p+1}w^{p+1}\Big|_{T_1}^{T_2}\\
&\quad + \int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2} - \int_{\mathbb{S}^1}\frac{(\partial_\theta w)^2}{2}\Big|_{T_1}^{T_2} - \frac{\beta(\beta-2)}{2}\int_{\mathbb{S}^1}\frac{w^2}{2}\Big|_{T_1}^{T_2}.
\end{aligned}$$

Since $\partial_t w = -\frac{\partial_r v}{r^{\beta-1}} + \beta\frac{v}{r^\beta}$, the last inequality then reads

$$\int_{T_1}^{T_2}\int_{\mathbb{S}^1}\left(\beta\frac{v}{r^\beta} - \frac{\partial_r v}{r^{\beta-1}}\right)^2 dt\,d\theta \le \widetilde{C},$$

where $\widetilde{C}$ depends only on the the constant M in the growth estimate $v(x) \le M|x|^\beta$, see Theorem 1.2. Since T_1 and T_2 are arbitrary, by the change of variable $r := e^{-t}$ we obtain that

$$\int_0^{1/2}\int_{\mathbb{S}^1}\left(\beta\,\frac{v(r,\theta)}{r^\beta} - \frac{\partial_r v(r,\theta)}{r^{\beta-1}}\right)^2 \frac{dr\,d\theta}{r} \le \widetilde{C}.$$

This implies the desired result via polar coordinates. □

From Theorem 1.3, we obtain the homogeneity of the blow-up sequences, according to Theorem 1.4:

Proof of Theorem 1.4. By (1.7), a change of variable $x = \rho y$ gives that

$$\int_{B_{\frac{1}{2\rho}}}\left(\beta\,\frac{v_\rho(y)}{|y|^\beta} - \frac{\partial_r v_\rho(y)}{|y|^{\beta-1}}\right)^2 \frac{dy}{|y|^2} \le \widetilde{C},$$

where the notation in (4.1) has been used. This and (4.2) imply that

$$\int_{\mathbb{R}^n}\left(\beta\,\frac{v_0(y)}{|y|^\beta} - \frac{\partial_r v_0(y)}{|y|^{\beta-1}}\right)^2 \frac{dy}{|y|^2} \le \widetilde{C},$$

and so

$$\beta\,\frac{v_0(y)}{|y|^\beta} = \frac{\partial_r v_0(y)}{|y|^{\beta-1}},$$

for any $y \in \mathbb{R}^n$, which implies the desired result (see e.g. Lemma 4.2 in [DSV15]). $\square$

4.2. **n-dimensional problems.** For the sake of completeness, we consider now a multidimensional model. We take

$$a_{ij}(x) := \delta_{ij} + \varepsilon x_i x_j |x|^{-2}. \tag{4.9}$$

Notice that the hypotheses in **(H1)-(H3)** are satisfied for sufficiently small $|\varepsilon|$.

We extend Theorem 1.4 to this case. To this aim, let us switch to polar coordinates and define

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ &\vdots \\ x_k &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{k-1}\cos\theta_k \\ &\vdots \\ x_n &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_n, \end{aligned}$$

where $0 \le \theta_k \le \pi$, with $k = 1, \ldots, n-2$, and $-\pi \le \theta_{n-1} \le \pi$. In this setting, the analogue of Lemma 4.1 goes as follows:

Lemma 4.2. *Let $\mathcal{L}$ be as in (1.1), with a_{ij} as in (4.9). Assume that x lies on the x_1 axis. Then*

$$\mathcal{L}v = (1+\varepsilon)\partial_{rr}v + \frac{1}{r}\partial_r v + \frac{1}{r^2}\partial_{\theta\theta}v. \tag{4.10}$$

Proof. From the chain rule, we have that

$$\frac{\partial v}{\partial x_1} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial v}{\partial \theta_1}\frac{\partial \theta_1}{\partial x_1} = \frac{\partial v}{\partial r}\cos\theta_1 - \frac{\sin\theta_1}{r}\frac{\partial v}{\partial \theta_1}. \tag{4.11}$$

Hence, proceeding as in (4.5), and using $\theta = 0$ to set the point on the x_1 axis, we get that

$$\frac{x_1^2}{|x|^2}\partial_{11}v = \partial_{rr}v,$$

which gives the desired result. $\square$

In this setting, the analogue of Theorem 1.4 is the following:

Theorem 4.3. *Let v be a strong solution of (1.1) in $B_1 \subset \mathbb{R}^n$ with a_{ij} as in (4.9). Assume that $0 \in \partial\{v > 0\}$ and v is non-degenerate at 0. Then any blow-up sequence at 0 has a converging subsequence such that the limit is a homogeneous function of degree $\beta = \frac{2}{1-p}$.*

Proof. We use the change of variables $r = e^{-t}, (\theta_1, \ldots, \theta_{n-1}) \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Hence, for the function $w(t,\theta) = \frac{v(r,\theta)}{r^\beta}$, making use of (4.10), equation (1.1) can be rewritten as

$$(1+\varepsilon)(\partial_{tt}w - \partial_t w - \beta(\beta-1)w) + (\beta w - \partial_t w + \Delta_{\theta\theta}w) = w^p,$$

where $\Delta_{\theta\theta}$ is the Laplace-Beltrami operator on the unit sphere. Thus, repeating the integration by parts as in the proof of Theorem 1.3 and the scaling argument in the proof of Theorem 1.4, the desired result follows. $\square$

5. Global homogeneous solutions

In this section, we would like to classify the global solutions of (1.1) in the plane in the homogeneous setting for the case of the obstacle problem.

Theorem 5.1. *Let $n = 2$, $\mathcal{L}$ be as in (1.1) and a_{ij} as in (1.4). Let v be a solution of (1.1) in $\mathbb{R}^n$ with $p = 0$ which is homogeneous of degree 2. Assume that $0 \in \partial\{v > 0\}$ and that $\partial\{v > 0\}$ is differentiable. Then ε in a_{ij} needs to be equal to 0 (and thus $a_{ij} = \delta_{ij}$).*

Proof. We first make a general calculation valid for all $p \in [0,1)$. Let $v(x) = r^\beta g(\theta)$. We suppose (up to a rotation) that the arc $(0, \alpha)$ is a component of the positivity set of g. In this way,

$$g(0) = g(\alpha) = 0. \tag{5.1}$$

We let $x_0 := (1,0)$. From Remark 3.2, we know that (3.1) is satisfied, and thus there exists $M > 0$ such that

$$M\,|x - x_0|^\beta \geq v(x) = r^\beta g(\theta) = r^\beta \big|g(\theta) - g(0)\big|.$$

For a small $t > 0$, we evaluate this formula at the point $x_t := (1,t)$, which corresponds in polar coordinate to $r_t := \sqrt{1+t^2}$ and $\theta_t = \arctan t$. In this way, we obtain that

$$Mt^\beta \geq (1+t^2)^{\frac{\beta}{2}} \big|g(\arctan t) - g(0)\big|.$$

So, dividing by t and sending $t \to 0$, using the fact that $\beta > 1$,

$$0 \geq \lim_{t\to 0} \left| \frac{g(\arctan t) - g(0)}{t} \right| = |g'(0)|$$

and so

$$g'(0) = 0. \tag{5.2}$$

Furthermore, from (4.3),

$$\beta(\beta-1)g + \beta(1+\varepsilon)g + (1+\varepsilon)g'' = g^p,$$

or equivalently

$$\beta(\beta+\varepsilon)g + (1+\varepsilon)g'' = g^p.$$

Multiplying both sides by g' and integrating yields

$$(1+\varepsilon)[g']^2 + \beta(\varepsilon+\beta)g^2 + C_o = \frac{2}{p+1} g^{p+1} \tag{5.3}$$

where $C_o \in \mathbb{R}$ is an arbitrary constant. Using (5.1) and (5.2), we have that $g(0) = 0 = g'(0)$, which gives that $C_o = 0$. Moreover

$$g^2 \left(\frac{g^{p-1}}{p+1} - \frac{\beta(\beta+\varepsilon)}{2} \right) \geq 0.$$

Consequently, solving (5.3) we obtain

$$g' = \pm \frac{1}{\sqrt{1+\varepsilon}} \sqrt{\frac{2}{p+1} g^{p+1} - \beta(\varepsilon+\beta)g^2}.$$

This is a separable equation, and so we obtain

$$\int \frac{dg}{\sqrt{\frac{2}{p+1} g^{p+1} - \beta(\varepsilon+\beta)g^2}} = \pm \frac{1}{\sqrt{1+\varepsilon}} \int d\theta + C. \tag{5.4}$$

The integrals above may be explicitly computed in terms of hypergeometric functions for any $p \in [0,1)$, but, for concreteness, we now restrict ourselves to the case $p = 0$. In this case, (5.4) becomes

$$\frac{1}{\sqrt{2}} \int \frac{dg}{\sqrt{g - (2+\varepsilon)g^2}} = \pm \frac{1}{\sqrt{1+\varepsilon}} \int d\theta + C. \tag{5.5}$$

We now set $a_\varepsilon := \frac{1}{2(2+\varepsilon)}$ and we observe that

$$g - (2+\varepsilon)g^2 = (2+\varepsilon)(2a_\varepsilon g - g^2) = (2+\varepsilon)(a_\varepsilon^2 - (a_\varepsilon - g)^2).$$

Hence, the substitution $h := (g/a_\varepsilon) - 1$ in (5.5) gives that

$$\frac{1}{\sqrt{2(2+\varepsilon)}} \int \frac{dh}{\sqrt{1-h^2}} = \pm \frac{1}{\sqrt{1+\varepsilon}} \int d\theta + C,$$

and so

$$\begin{aligned} \frac{1}{\sqrt{2(2+\varepsilon)}} \arcsin \frac{g - a_\varepsilon}{a_\varepsilon} &= \frac{1}{\sqrt{2(2+\varepsilon)}} \arcsin h \\ &= \pm \frac{1}{\sqrt{1+\varepsilon}} \int d\theta + C \\ &= \pm \frac{1}{\sqrt{1+\varepsilon}}\, \theta + C. \end{aligned} \tag{5.6}$$

Then, evaluating (5.6) at $\theta := 0$ and using (5.1), we obtain that

$$\arcsin(-1) = \arcsin\frac{g(0) - a_\varepsilon}{a_\varepsilon} = \sqrt{2(2+\varepsilon)}\, C.$$

Thus, defining

$$\omega_\varepsilon := \pm\sqrt{\frac{2(2+\varepsilon)}{1+\varepsilon}},$$

we rewrite (5.6) as

$$\arcsin\frac{g(\theta) - a_\varepsilon}{a_\varepsilon} = \omega_\varepsilon\theta + \arcsin(-1). \tag{5.7}$$

Since $\partial\{v > 0\}$ is smooth and v homogeneous, formula (5.1) says that $\alpha = k\pi$, with $k \in \{1, 2\}$. Evaluating (5.7) at $\theta := k\pi$ and $\theta := 0$, using that $g(0) = g(k\pi) = 0$ (in view of (5.1)), we obtain that

$$\begin{aligned} 0 &= \frac{g(k\pi) - a_\varepsilon}{a_\varepsilon} - \frac{g(0) - a_\varepsilon}{a_\varepsilon} \\ &= \sin\left(\omega_\varepsilon\, k\pi + \arcsin(-1)\right) - \sin\left(\arcsin(-1)\right) \\ &= -\cos\left(\omega_\varepsilon\, k\pi\right) + 1 \end{aligned}$$

and therefore $\omega_\varepsilon\, k\pi \in 2\pi\mathbb{Z}$. This gives that

$$\pm k\sqrt{\frac{2(2+\varepsilon)}{1+\varepsilon}} \in 2\mathbb{Z},$$

and so

$$\sqrt{\frac{2(2+\varepsilon)}{1+\varepsilon}} \in \mathbb{Z},$$

which, for small ε, only holds when $\varepsilon = 0$. $\square$

Remark 5.2. From (5.7), one can also construct a homogeneous solution $v \geq 0$ of the obstacle problem $\mathcal{L}v = 1$ in $\{v > 0\}$, with $\mathcal{L}$ as in (1.1) and a_{ij} as in (1.4), whose free boundary is a cone, namely, in polar coordinates, one can take $v = v(r, \theta) = r^2\, g(\theta)$, with

$$g(\theta) = \begin{cases} a_\varepsilon\left(1 - \cos(\omega_\varepsilon\theta)\right) & \text{if } \theta \in \left(0, \frac{2\pi}{\omega_\varepsilon}\right), \\ 0 & \text{otherwise,} \end{cases}$$

where $a_\varepsilon := \frac{1}{2(2+\varepsilon)}$ and $\omega_\varepsilon := \sqrt{\frac{2(2+\varepsilon)}{1+\varepsilon}} < 2$ when $\varepsilon > 0$ (respectively, $\omega_\varepsilon := \sqrt{\frac{2(2+\varepsilon)}{1+\varepsilon}} > 2$ when $\varepsilon < 0$), see Figure 1. Notice in particular, that the singular cone of the free boundary can be either obtuse or acute, according to the cases $\varepsilon > 0$ and $\varepsilon < 0$.

Theorem 1.5 says that this example is somehow “typical”, namely if the free boundary of (1.1) meets the discontinuity points of the coefficients a_{ij} in a non-degenerate way, then a singularity occurs. The proof of this fact is based on Theorem 5.1, and the details go as follows:

Proof of Theorem 1.5. Assume by contradiction that $\partial\{v > 0\}$ can be written as a differentiable graph near the origin: say, up to a rotation, that $\{v > 0\}$ coincides with $\{x_2 < \varphi(x_1)\}$ near the origin, with φ differentiable, $\varphi(0) = 0$ and $\varphi'(0) = 0$. We consider the blow-up sequence v_{r_k} as in (4.1) (with $x_0 = 0$). From the discussion at the beginning of Section 4, we know that, for a suitable infinitesimal sequence r_k, it holds that v_{r_k} approaches a global solution v_0. Near the origin, we have that $\partial\{v_{r_k} > 0\}$ coincides with $\left\{x_2 < \frac{\varphi(r_k x_1)}{r_k}\right\}$. Using this and the fact that $\varphi(r_k x_1) = o(r_k x_1)$, we thus obtain that $\partial\{v_0 > 0\}$ near the origin coincides with $\{x_2 < 0\}$. Also, from Theorem 1.4, we know that v_0 is homogeneous of degree 2. These considerations and Theorem 5.1 imply that $\varepsilon = 0$, against our assumptions. $\square$

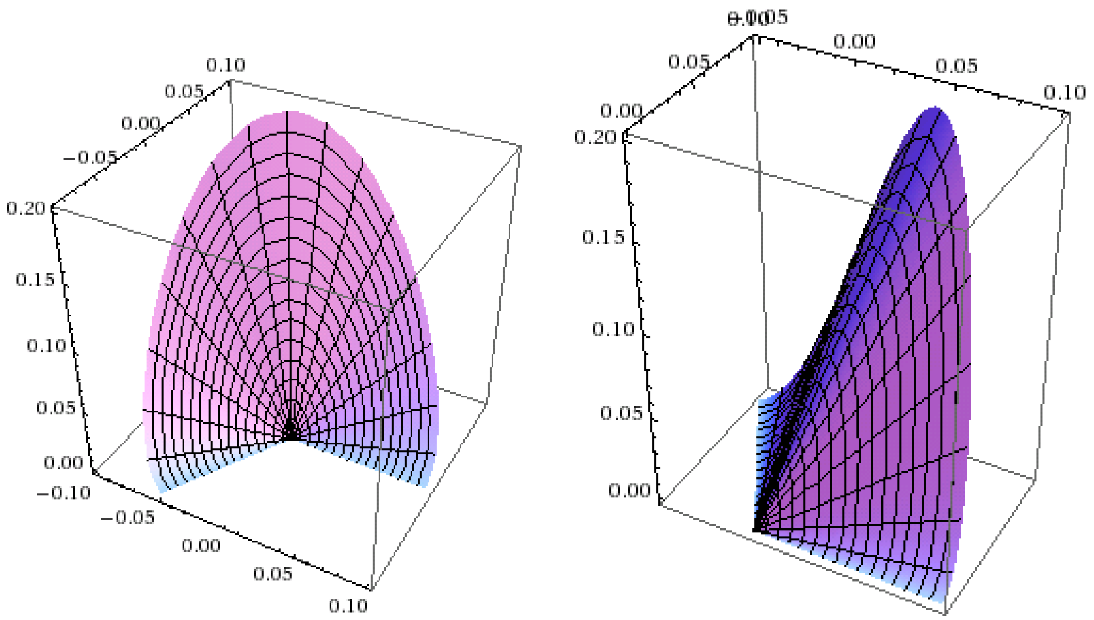

FIGURE 1. Examples of homogeneous solutions of the obstacle problem with obtuse/acute singular free boundary.

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(Serena Dipierro) School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia

E-mail address: s.dipierro@unimelb.edu.au

(Aram Karakhanyan) Maxwell Institute for Mathematical Sciences and School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom

E-mail address: aram.karakhanyan@ed.ac.uk

(Enrico Valdinoci) School of Mathematics and Statistics, University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia, and Istituto di Matematica Applicata e Tecnologie Informatiche, Consiglio Nazionale delle Ricerche, Via Ferrata 1, 27100 Pavia, Italy, and Weierstrass Institut für Angewandte Analysis und Stochastik, Mohrenstrasse 39, 10117 Berlin, Germany, and Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy

E-mail address: enrico@mat.uniroma3.it